M.Sc (Sem 03) Paper 11
Unit - 03

The closed graph theorem:-

Closed linear Transformation: Def:- Let N and N' be normed linear spaces and let D be a subspace of N. Then a linear transformation $T: D \to N'$ is said to be closed iff $x_n \in D$, $x_n \to x$ and $T(x_n) \to y$ imply $x \in D$ and $y = T(x)$.

Statement:- Let B and B' be Banach spaces and let T be a linear transformation of B onto B'. Then T is a continuous mapping if and only if its graph is closed.
In other words - If X and Y are Banach spaces and if T is a linear operator of X into Y, then T is continuous iff its graph G_T is closed.

Proof:- The 'only if' part, let T be continuous and let T_G be the graph of T. We shall show that $\bar{T} = T_G$ and this will prove that T_G is closed.
Since $T_G \subset \bar{T}_G$ always, we need only prove $\bar{T}_G \subset T_G$.
So let $(x, y) \in \bar{T}_G$.
Then (x, y) is an adherent point of T_G.
Hence there exists a sequence $\langle x_n, T(x_n) \rangle$ in T_G such that $(x_n, T(x_n)) \to (x, y)$ which implies that $x_n \to x$ and $T(x_n) \to y$. But, since T is continuous, $x_n \to x$ implies $T(x_n) \to T(x)$ and so $y = T(x)$. This shows that $(x, y) = (x, T(x)) \in T_G$ and so $\bar{T}_G \subset T_G$.
The 'if' part. Let T_G ~~be~~ is closed. We denote by B_1 the linear space B renormed by

$$\|x\|_1 = \|x\| + \|T(x)\|.$$

Now $\|T(x)\| \leq \|x\| + \|T(x)\| = \|x\|$.
Which shows that T is bounded and consequently continuous.
It therefore suffices to show that B and B_1 have the same topology, that is, they are homeomorphic. Consider the identity map $I: B_1 \to B$; $I(x) = x \ \forall \ x \in B_1$.
Clearly I is one-one onto. Further

$$\|I(x)\| = \|x\| \leq \|x\| + \|T(x)\| = \|x\|_1.$$

Which shows that I is bounded and hence continuous.
If we can show that B_1 is complete, then ~~we will guarantee~~

We have I is a homeomorphism and thus the config... to prove. So let $\langle x_n \rangle$ be any Cauchy sequence in B_1

$$\|x_n - x_m\|_1 \to 0 \text{ as } m, n \to \infty$$

$$\Rightarrow \|x_n - x_m\| + \|T(x_n - x_m)\| \to 0 \text{ as } m, n \to \infty$$

$$\Rightarrow \|x_n - x_m\| + \|T(x_n) - T(x_m)\| \to 0 \text{ as } m, n \to \infty$$

$$\Rightarrow \|x_n - x_m\| \to 0 \text{ and } \|T(x_n) - T(x_m)\| \to 0$$

$$\text{as } m, n \to \infty$$

$\Rightarrow \langle x_n \rangle$ is a Cauchy sequence in B and $\langle T(x_n) \rangle$ is a Cauchy sequence in B' ——— (1)

Since the graph T_G is given to be closed (1) shows that $(x, y) \in T_G$ so that $y = T(x)$.

Now
$$\|x_n - x\|_1 = \|x_n - x\| + \|T(x_n - x)\|$$

$$= \|x_n - x\| + \|T(x_n) - T(x)\|$$

$$= \|x_n - x\| + \|T(x_n) - y\| \quad [\because y = T(x)]$$

$$[\because x_n \to x \text{ and } T(x_n) \to y]$$

It follows that the sequence $\langle x_n \rangle$ in B_1 converges to $x \in B_1$ and hence B_1 is complete as required.

[Note that B_1 and B are the same sets so that $x \in B \Rightarrow x \in B_1$].

Proved.

Theorem: — Let N and N' be normed linear spaces and D a subspace of N. Then a linear transformation $T: D \to N'$ is closed if and only if its graph T_G is closed.

Proof: — First suppose T is closed linear transformation. To show that its graph T_G is closed, i.e. T_G contains all its limit points.

Let (x, y) be any limit point of T_G. Then there exists a sequence of points in T_G, $\langle x_n, T(x_n) \rangle$, where $x_n \in D$, converging to (x, y).

But
$$\langle x_n, T(x_n) \rangle \to (x, y)$$

$$\Rightarrow \|(x_n, T(x_n)) - (x, y)\| \to 0$$

$$\Rightarrow \|(x_n - x), T(x_n) - y)\| \to 0$$

$$\Rightarrow \|x_n - x\| + \|T(x_n) - y\| \to 0$$

$$\Rightarrow \|x_n - x\| \to 0 \text{ and } \|T(x_n) - y\| \to 0$$

$$\Rightarrow x_n \to x \text{ and } T(x_n) \to y$$

$\Rightarrow x \in D$ and $T(x) = y$ ($\because$ T is closed by ~~defi~~ above defi.)

$\Rightarrow (x, y) \in T_4$ by definition of graph.

Thus we have shown that every limit point of T_4 is in T_4 and so T_4 is closed.

Conversely let the graph T_4 of T be closed.

To show that T is a closed linear transformation

let $x_n \in D$, $x_n \to x$ and $T(x_n) \to y$.

Then it is easy to see that (x, y) is an adherent point of T_4 so that $(x, y) \in \overline{T_4}$. But $\overline{T_4} = T(4)$ since T_4 is given to be closed. Hence $(x, y) \in T_4$ and so by definition of T_4, we have $x \in D$ and $y = T(x)$.

Therefore T is a closed linear transformation.

Anjani Kumar Singh.